The Chinese University of Hong Kong
Department of Mathematics
2024 Enrichment Programme for Young Mathematics Talents

TOWARDS DIFFERENTIAL GEOMETRY
Test 2, 23/08/2024

Solution

Instructions:

- Time allowed: $\mathbf{90 \pm \delta}$ **minutes.**
- This paper consists of **Basic Part**, **Harder Part** and **Bonus Part**.
- The full mark of the paper is **80 points** and bonus mark **15 points**.
- Answer **ALL** questions in Basic Part and **THREE** questions in Harder Part. Make your best effort to answer the Bonus Part.
- Show your work clearly and concisely. Give thorough explanation and justification for your calculations and observations.
- Write your answers in the spaces provided in the Answer Booklet. Begin each question on a new page. Clearly indicate the question number in the designated slot at the top of each page.
- Supplementary answer sheets and rough paper will be supplied on request.
- Non-graphical calculators are allowed.
- Unless otherwise specified, numerical answers must be exact.

Full Name: ______________________________ Group: ____________________

Basic Part (50 points). Answer **ALL** questions in this part.

1. (10 points) Compute the curvature $\kappa(t)$ of the following curves:

   (a) (4 points) $\alpha(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$, for $t \in (0,1)$.

   (b) (3 points) $\alpha(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$, for $t \in (0, 2\pi)$.

   (c) (3 points) $\alpha(t) = (t, \cosh t)$, for $t \in (0, \infty)$.

   Solution.

   (a) We have

$$\begin{aligned}
\alpha(t) &= \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)\\
\alpha'(t) &= \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right)\\
||\alpha'(t)|| &= \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}} = 1.
\end{aligned}$$

   Therefore, α is parameterized by arc-length and we have $\mathbf{T}(t) = \alpha'(t)$ and

$$\mathbf{T}'(t) = \left(\frac{1}{4\sqrt{1+t}}, \frac{1}{4\sqrt{1-t}}, 0\right).$$

   Thus, $\kappa(t) = ||\,\mathbf{T}'(t)|| = \frac{1}{4}\sqrt{\frac{1}{1+t} + \frac{1}{1-t}} = \frac{1}{4}\sqrt{\frac{2}{1-t^2}} = \frac{\sqrt{2}}{4\sqrt{1-t^2}}$.

   (b) We have

$$\begin{aligned}
\alpha(t) &= \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)\\
\alpha'(t) &= \left(-\frac{4}{5}\sin t, -\cos t, \frac{3}{5}\sin t\right)\\
||\alpha'(t)|| &= \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t} = \sqrt{\sin^2 t + \cos^2 t} = 1.
\end{aligned}$$

   Therefore, α is parameterized by arc-length and we have $\mathbf{T}(t) = \alpha'(t)$ and

$$\mathbf{T}'(t) = \left(-\frac{4}{5}\cos t, \sin t, \frac{3}{5}\cos t\right).$$

   Thus, $\kappa(t) = ||\,\mathbf{T}'(t)|| = \sqrt{\frac{16}{25}\cos^2 t + \sin^2 t + \frac{9}{25}\cos^2 t} = \sqrt{\sin^2 t + \cos^2 t} = 1.$

(c) We have

$$\alpha(t) = (t, \cosh t)$$
$$\alpha'(t) = (1, \sinh t)$$
$$||\alpha'(t)|| = \sqrt{1+\sinh^2 t} = \sqrt{\cosh^2 t}.$$

Note that since $\cosh t \geq 0$ for any $t \in (0.\infty)$, we have

$$\begin{aligned} ||\alpha'(t)|| &= \cosh t \\ \mathbf{T}(t) &= \frac{\alpha'(t)}{||\alpha'(t)||} = (\operatorname{sech} t, \tanh t) \\ \mathbf{T}'(t) &= \left(-\tanh t \operatorname{sech} t, \operatorname{sech}^2 t\right) \\ ||\mathbf{T}'(t)|| &= \operatorname{sech} t\sqrt{\tanh^2 t + \operatorname{sech}^2 t} = \operatorname{sech} t \\ \kappa(t) &= \frac{||\mathbf{T}'(t)||}{||\alpha'(t)||} = \operatorname{sech}^2 t. \end{aligned}$$

Candidate Performance Report

Most candidates performed well. Many candidates proceed by computing unit tangent vector, then divide by $\|\mathbf{r}'(t)\|$ to compute curvature. Over 70% candidates noticed (a) and (b) are parametrized by arc-length, which heavily simplifies the computation. However in (c), a substantial number of candidates did not notice that $\alpha(t) = (t, \cosh t)$ is not parametrized by arc-length. Therefore they mistakenly compute $\kappa(t) = \|\mathbf{r}''(t)\|$. It is worth to note that, on one hand, one candidate identified the graphical function $f(t) = \cosh t$ and computed curvature by $\kappa(t) = \dfrac{|f''(t)|}{\left(1+(f'(t))^2\right)^{3/2}}$, which is appreciated. On the other hand, some candidates computed curvature by $\kappa(t) = \dfrac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$ in (a), (b), resulting in a large amount of computation mistakes and loss of time.

2. (20 points) Given the parameterized curve

$$\mathbf{r}(t) = \left(e^t \cos t, e^t \sin t, 2e^t\right), \quad t \in (0, \infty).$$

   (a) (5 points) Find $\mathbf{r}'(t)$, hence find $\|\mathbf{r}'(t)\|$.

   (b) (2 points) Find $s := \int_0^t \|\mathbf{r}'(u)\| du$.

   (c) (3 points) Hence, or otherwise, find an arc-length parameterization of $\mathbf{r}(t)$. Please state clearly the domain of s.

   (d) (10 points) Hence, or otherwise find the **TNB** frame, the curvature κ, and torsion τ. (You may find your answer with the original parameterization, or with the arc-length parameterization you found.)

Solution.

In this solution, we will repeatedly use the fact that

$$(\cos\theta - \sin\theta)^2 + (\sin\theta + \cos\theta)^2 = \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = 2$$

for any $\theta \in \mathbb{R}$.

   (a) By product rule, $\mathbf{r}'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 2e^t)$. Thus

$$||\mathbf{r}'(t)|| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 4} = e^t\sqrt{.}$$

   (b) Integrating, we have $s = \int_0^t e^u \sqrt{6} du = e^t\sqrt{6} - \sqrt{6}$.

   (c) By (b), we have $s = \sqrt{6}(e^t - 1)$, thus $t = \log\left(\frac{s}{\sqrt{6}} + 1\right)$. Therefore,

$$\mathbf{r}(s) = \left(\left(\frac{s}{\sqrt{6}} + 1\right)\cos\left(\log\left(\frac{s}{\sqrt{6}} + 1\right)\right), \left(\frac{s}{\sqrt{6}} + 1\right)\sin\left(\log\left(\frac{s}{\sqrt{6}} + 1\right)\right), \left(\frac{s}{\sqrt{6}} + 1\right)\right).$$

   Note that as $t \to 0$, $s \to 0$, and as $t \to \infty$, $s \to \infty$. Thus $s \in (0, \infty)$.

(d) Using the given parameterization, we have

$$
\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{||\,\mathbf{r}'(t)||} = \frac{1}{\sqrt{6}}\,(\cos t - \sin t, \cos t + \sin t, 2) \\
\mathbf{T}'(t) &= \frac{1}{\sqrt{6}}\,(-\sin t - \cos t, -\sin t + \cos t, 0) \\
||\,\mathbf{T}'(t)|| &= \frac{1}{\sqrt{6}}\sqrt{(-\sin t - \cos t)^2 + (-\sin t + \cos t)^2} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}} \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{||\,\mathbf{T}'(t)||} = \frac{1}{\sqrt{2}}\,(-\sin t - \cos t, -\sin t + \cos t, 0) \\
\mathbf{N}'(t) &= \frac{1}{\sqrt{2}}\,(-\cos t + \sin t, -\cos t - \sin t, 0) \\
\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) &= \frac{1}{2\sqrt{3}}\big(2(\sin t - \cos t), -2(\sin t + \cos t), (\cos t - \sin t)^2 + (\cos t + \sin t)^2\big) \\
&= \frac{1}{2\sqrt{3}}\big(2(\sin t - \cos t), -2(\sin t + \cos t), 2\big) \\
&= \frac{1}{\sqrt{3}}\,(\sin t - \cos t, -\sin t - \cos t, 1) \\
\kappa &= \frac{||\,\mathbf{T}'(t)||}{||\,\mathbf{r}'(t)||} = \frac{1}{3e^t\sqrt{2}} \\
\tau &= \frac{1}{||\,\mathbf{r}'(t)||}\langle \mathbf{N}'(t), \mathbf{B}(t)\rangle \\
&= \frac{1}{e^t\sqrt{6}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}}\,\big((\sin t - \cos t)^2 + (-\sin t - \cos t)^2\big) = \frac{1}{6e^t}(2) = \frac{1}{3e^t}.
\end{aligned}
$$

Alternative solution using arc-length parameterization:

Let $\alpha := \left(\frac{s}{\sqrt{6}} + 1\right)$, and $(\cdot)'$ denotes derivative with respect to s. Note that

$$\frac{d}{ds} = \frac{d\alpha}{ds}\frac{d}{d\alpha} = \frac{1}{\sqrt{6}}\frac{d}{d\alpha}$$

. With such notation, $\mathbf{r}(s) = \big(\alpha\cos(\log\alpha), \alpha\sin(\log\alpha), \alpha\big)$. We then have

$$\begin{aligned}
\mathbf{r}'(s) &= \frac{1}{\sqrt{6}}\big(\cos(\log\alpha) - \alpha\sin(\log\alpha)\frac{1}{\alpha}, \sin(\log\alpha) + \alpha\cos(\log\alpha)\frac{1}{\alpha}, 1\big) \\
&= \frac{1}{\sqrt{6}}\big(\cos(\log\alpha) - \sin(\log\alpha), \sin(\log\alpha) + \cos(\log\alpha), 1\big) \\
\mathbf{T}(s) = \mathbf{r}(s) &= \frac{1}{\sqrt{6}}\big(\cos(\log\alpha) - \sin(\log\alpha), \sin(\log\alpha) + \cos(\log\alpha), 1\big) \\
\mathbf{T}'(s) &= \frac{1}{6}\left(-\frac{\sin(\log\alpha)}{\alpha} - \frac{\cos(\log\alpha)}{\alpha}, \frac{\cos(\log\alpha)}{\alpha} - \frac{\sin(\log\alpha)}{\alpha}, 0\right) \\
||\,\mathbf{T}'(s)|| &= \frac{1}{6\alpha}\sqrt{(-\sin(\log\alpha) - \cos(\log\alpha))^2 + (\cos(\log\alpha) - \sin(\log\alpha))^2} \\
&= \frac{1}{6\alpha}\sqrt{2} = \frac{\sqrt{2}}{6\alpha} \\
\mathbf{N}(s) &= \frac{\mathbf{T}'(s)}{||\,\mathbf{T}'(s)||} = \frac{1}{\sqrt{2}}(-\sin(\log\alpha) - \cos(\log\alpha), \cos(\log\alpha) - \sin(\log\alpha), 0) \\
\mathbf{N}'(s) &= \frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}\left(-\frac{\cos(\log\alpha)}{\alpha} + \frac{\sin(\log\alpha)}{\alpha}, -\frac{\sin(\log\alpha)}{\alpha} - \frac{\cos(\log\alpha)}{\alpha}, 0\right) \\
&= \frac{1}{2\sqrt{3}}\left(-\frac{\cos(\log\alpha)}{\alpha} + \frac{\sin(\log\alpha)}{\alpha}, -\frac{\sin(\log\alpha)}{\alpha} - \frac{\cos(\log\alpha)}{\alpha}, 0\right) \\
\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) &= \frac{1}{2\sqrt{3}}\big(2(\sin(\log\alpha) - \cos(\log\alpha)), -2(\sin(\log\alpha) + \cos(\log\alpha)), 2\big) \\
&= \frac{1}{\sqrt{3}}\big(\sin(\log\alpha) - \cos(\log\alpha), -\sin(\log\alpha) - \cos(\log\alpha), 1\big) \\
\kappa(s) = ||\,\mathbf{T}'(s)|| &= \frac{\sqrt{2}}{6\alpha} = \frac{\sqrt{2}}{6\left(\frac{s}{\sqrt{6}} + 1\right)} = \frac{\sqrt{3}}{3s + 3\sqrt{6}} \\
\tau(s) &= \langle \mathbf{N}'(s), \mathbf{B}(s)\rangle \\
&= \frac{1}{6\alpha}\left((-\sin(\log\alpha) - \cos(\log\alpha))^2 + (\sin(\log\alpha) - \cos(\log\alpha))^2\right) \\
&= \frac{1}{6\alpha}(2) = \frac{1}{3\alpha} = \frac{1}{3\left(\frac{s}{\sqrt{6}} + 1\right)} = \frac{\sqrt{6}}{3s + 3\sqrt{6}}
\end{aligned}$$

Candidate Performance Report

This question investigates the Frenet Frame of $\mathbf{r}(t) = (e^t \cos t, e^t \sin t, 2e^t)$.

The first 10 marks examines derivation of arc-length parametrization of $\mathbf{r}(t)$. Students are required to integrate $\int_0^t \sqrt{6}e^u du$. Notably many candidates are not aware of $e^0 = 1$ but mistakenly wrote $e^0 = 0$. So they give $t = \ln\left(\frac{s}{\sqrt{6}}\right)$ as answer, instead of $t = \ln\left(\frac{s}{\sqrt{6}} + 1\right)$. For arc-length parametrization (unique up to constant), we accept both. Unfortunately, there are candidates unfamiliar with product rule and mistakenly write $\mathbf{r}'(t) = (-e^t \sin t, e^t \cos t, 2e^t)$ and bring the wrong answer to all subsequent parts. Since Q2 is heavily scaled up to give basic scores to candidates, those candidates suffered from an enormous loss of score.

It is also good to see most candidates included the range of parameter s after parametrizing $\mathbf{r}$ by arc-length. One candidate justified the range with rigorous limit, which is appreciated. He wrote: Since e^t is an increasing function for $t \in (0, \infty)$, hence $\lim_{t\to 0^+} \sqrt{6}(e^t - 1) = 0$. and $\lim_{t\to 0^+} \sqrt{6}(e^t - 1) = 0$, hence the domain of $s = (0, \infty)$

Regarding the second 10 marks, candidates are required to find $\mathbf{T}, \mathbf{N}, \mathbf{B}$ frame, curvature κ and torsion τ. Most candidates approached the question using the original parametrization while there are around 10 candidates used arc-length parametrization. Statistically, no any candidate score full mark in (d) with arc-length parametrization. It is attributed to large amount of calculation mistakes on complex functional form of $\mathbf{r}(s)$. The chief examiner of question 2 commented "When you see it being so ugly you should know it is a trap la... At least make a substitution to make your life better. Totally doable in less than 10mins though."

One common mistake is: When computing $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, many candidates forgot the negative sign of the $\mathbf{j}$ component in the cross product, in this test, we give a mark whenever you write the formula (if used correctly). Please be careful especially when participating in a public exam. Also, a useful technique is to **factor out the scalar** before doing inner product, cross product or taking norm.

Another common mistake for those approached (d) using original parametrization is: They forget to divide by $\|\mathbf{r}'(t)\|$ in computation of $\tau(t)$ using the formula: $\tau(t) = \left\langle \frac{\mathbf{N}'(t)}{\|\mathbf{r}'(t)\|}, \mathbf{B}(t) \right\rangle$. While examiners appreciate most candidates can memorise formulae: $\tau = \left\langle \frac{d\mathbf{N}}{ds}, \mathbf{B} \right\rangle$, condition / preprequistite of the formula is equally important. And, yes, all formulae are provided in formula sheet.

3. (20 points) It is given that the parametrization of the torus is

$$\mathbf{X}(\phi,\theta) = ((R + r\sin\phi)\cos\theta, (R + r\sin\phi)\sin\theta, r\cos\phi), 0 < \phi < 2\pi, 0 < \theta < 2\pi$$

where $R > r > 0$ and R, r are constants.

(a) (6 points) Compute $\mathbf{X}_\phi \times \mathbf{X}_\theta$.

(b) (4 points) Hence, prove that the torus is a regular surface.

(c) (5 points) Compute the first fundamental form of $\mathbf{X}$ as a 2×2 matrix.

(d) (5 points) Show that the surface area of the torus is $4\pi^2 rR$.

Solution.

(a)

$$\begin{aligned}
\mathbf{X}_\phi =& \big(r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi\big)\\
\mathbf{X}_\theta =& \big(-(R + r\sin\phi)\sin\theta, (R + r\sin\phi)\cos\theta, 0\big)\\
\mathbf{X}_\phi \times \mathbf{X}_\theta =& \big(r(R + r\sin\phi)\sin\phi\cos\theta, r(R + r\sin\phi)\sin\theta\sin\phi, r(R + r\sin\phi)\cos\phi\big)
\end{aligned}$$

(b) Suppose $\mathbf{X}_\phi \times \mathbf{X}_\theta = (0, 0, 0)$, then $r(R + \sin\phi)\cos\phi = 0$, we have $r = 0$ or $r\sin\phi = 0$ or $\cos\phi = 0$. As $r > 0$ and $R > r \geq r\sin\phi$, we have $r \neq 0$ and $R + r\sin\phi \neq 0$. Suppose $\cos\phi = 0$, then $\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We have $r(R + r\sin\phi)\sin\phi\cos\theta = 0$ iff $\cos\theta = 0$ iff $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. But, $r(R + \sin\phi)\sin\phi\cos\theta \neq 0$. Contradiction arises.

(c)

$$I = \begin{bmatrix} r^2 & 0 \\ 0 & (R + r\sin\phi)^2 \end{bmatrix}$$

(d)

$$\begin{aligned}
\text{Area} =& \int_0^{2\pi}\int_0^{2\pi} \sqrt{\det I}\, d\theta d\phi\\
=& \int_0^{2\pi}\int_0^{2\pi} \sqrt{r^2(R + r\sin\phi)^2}\, d\theta d\phi\\
=& \int_0^{2\pi}\int_0^{2\pi} r(R + r\sin\phi)\, d\theta d\phi \qquad\qquad \text{As } r(R + r\sin\phi) \geq 0\\
=& 2\pi\int_0^{2\pi} (rR + Rr\sin\phi)\, d\phi\\
=& 2\pi(2\pi)rR\\
=& 4\pi^2 rR
\end{aligned}$$

Candidate Performance Report

The performance of this question is very good. Many candidates scored 17-20 marks.

Notably there are very few candidates can get perfect score due to imperfect argument on regularity of torus. The reasoning of "$\sin\phi$ and $\cos\phi$ are not simultaneously zero" is often incomplete (didn't exhaust all possibilities).

One candidate approached this question elegantly: The candidate notice that it suffices to show that $\|\mathbf{X}_\phi \times \mathbf{X}_\theta\|^2$ is non-vanishing. He computed $\|\mathbf{X}_\phi \times \mathbf{X}_\theta\|^2$ indirectly, through vector identity in lecture notes prop 1.3.17 (2). He first computed E, F, G and then wrote $\|\mathbf{X}_\phi \times \mathbf{X}_\theta\|^2 = EG - F^2 = \det(I) = r(R + r\sin\phi) > 0$ since $R > r > 0$. This approach saved a lot of time and troubles on auguing "not simultaneously zero". There are also one candidate computed the differential matrix $D\,\mathbf{X}$ and hence find 1st fundamental form by $I = (D\,\mathbf{X})^T(D\,\mathbf{X})$. Such response derserves a honourable mention, for the effort and knowledge.

Considering the common mistakes, some candidates expanded $\mathbf{X}(\phi, \theta)$ as follows:

$$(R\cos\theta + r\sin\phi\cos\theta, R\sin\theta + r\sin\phi\sin\theta, r\cos\phi)$$

and then computed the cross product $\mathbf{X}_\phi \times \mathbf{X}_\theta$ and its norm as follows:

$$\big(rR\sin\phi\cos\theta + r^2\sin\phi\sin\phi\cos\theta, rR\sin\theta\sin\phi + r^2\sin\phi\sin\theta\sin\phi, rR\cos\phi + r^2\sin\phi\cos\phi\big)$$

$$\|\mathbf{X}_\phi \times \mathbf{X}_\theta\| = \sqrt{\begin{matrix} \big(rR\sin\phi\cos\theta + r^2\sin\phi\sin\phi\cos\theta\big)^2 \\ +\big(rR\sin\theta\sin\phi + r^2\sin\phi\sin\theta\sin\phi\big)^2 \\ +\big(rR\cos\phi + r^2\sin\phi\cos\phi\big)^2 \end{matrix}}$$

The computation is subsequently horribly tedious and impossible to proceed. Candidates are recommended to simplify their expressions before proceed. E.g. factor out common factors and adopt appropriate notations to make life easier. (Comparison: HKUST Professor Fredrick Fong spent 1 hour on this question)

Another common mistake is: candidates memorise the definition of 1st fundamental form incorrectly. Some candidates claim that:

$$I = \begin{pmatrix} \langle \mathbf{X}_{\theta\theta}, \mathbf{X}_{\theta\theta} \rangle & \langle \mathbf{X}_{\theta\theta}, \mathbf{X}_{\phi\phi} \rangle \\ \langle \mathbf{X}_{\phi\phi}, \mathbf{X}_{\theta\theta} \rangle & \langle \mathbf{X}_{\phi\phi}, \mathbf{X}_{\phi\phi} \rangle \end{pmatrix}$$

They computed first fundamental form, hence surface area of torus incorrectly. Since this question, formulated with good intentions, is heavily scaled up to enable all candidates to earn basic scores, the result can be disastrous.

Harder Part – Curve Theory (15 points).

Answer **ONE** question in **Curve Theory** section

Curve Theory – Structured Questions

4. (15 points) This question investigates vanishing curvature κ and torsion τ

   (a) (6 points) Let $\mathbf{r}(t)$ be a regular parametrised curve. Prove that if its curvature satisfies $\kappa(t) = 0$ of any $a < t < b$, then $\mathbf{r}(t)$ is a straight line.

   (b) (9 points) Let $\mathbf{r}(s)$ be a regular parametrised curve with $\| \mathbf{r}'(s)\| = 1$. Further suppose that there exists a constant $\mathbf{v} \in \mathbb{R}^3, C \in \mathbb{R}$ such that $\langle \mathbf{r}(s), \mathbf{v}\rangle = C$ for all s.

      (i) (4 points) Differentiating with respect to s, prove that

$$\langle \mathbf{r}'(s), \mathbf{v}\rangle = \langle \mathbf{r}''(s), \mathbf{v}\rangle = \langle \mathbf{r}'''(s), \mathbf{v}\rangle = 0.$$

      (ii) (3 points) Explain why $\langle \mathbf{r}'(s) \times \mathbf{r}''(s), \mathbf{r}'''(s)\rangle = 0$.

      (iii) (2 points) Compute torsion $\tau(s)$ of $\mathbf{r}(s)$.

Solution.

(a) Suppose $\kappa(t) = 0$, since $\kappa(t) = \dfrac{\| \mathbf{T}'(t)\|}{\| \mathbf{r}'(t)\|}$, hence $\| \mathbf{T}'(t)\| = 0$

Therefore we have $\mathbf{T}'(t) = \mathbf{0}$, integrating gives $\mathbf{T}(t) = \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^n$ constant

By definition of unit tangent vector, $\dfrac{\mathbf{r}'(t)}{\| \mathbf{r}'(t)\|} = \mathbf{a}$

Hence $\mathbf{r}'(t) = \| \mathbf{r}'(t)\| \, \mathbf{a}$

Integrating again, denote $I(t) = \int_a^t \| \mathbf{r}'(t)\| dt$, then $\mathbf{r}(t) = I(t)\,\mathbf{a} + \mathbf{b}$, for $a < t < b$

By regularity, $I'(t) = \| \mathbf{r}'(t)\| > 0$, hence $I : \mathbb{R} \to \mathbb{R}$ is strictly increasing

So we conclude that $\mathbf{r}(t) = I(t)\,\mathbf{a} + \mathbf{b}$ represents a straight-line

(b) (i). Differentiate Both Sides Repeatedly:

$$\begin{aligned}
\langle \mathbf{r}(s), \mathbf{v}\rangle &= C \\
\frac{d}{ds}\langle \mathbf{r}(s), \mathbf{v}\rangle &= 0 \\
\langle \mathbf{r}'(s), \mathbf{v}\rangle + \langle \mathbf{r}(s), \mathbf{v}'\rangle &= 0 \\
\langle \mathbf{r}'(s), \mathbf{v}\rangle &= 0 \\
\frac{d}{ds}\langle \mathbf{r}'(s), \mathbf{v}\rangle &= 0 \\
\langle \mathbf{r}''(s), \mathbf{v}\rangle + \langle \mathbf{r}'(s), \mathbf{v}'\rangle &= 0 \\
\langle \mathbf{r}''(s), \mathbf{v}\rangle &= 0 \\
\frac{d}{ds}\langle \mathbf{r}''(s), \mathbf{v}\rangle &= 0 \\
\langle \mathbf{r}'''(s), \mathbf{v}\rangle + \langle \mathbf{r}''(s), \mathbf{v}'\rangle &= 0 \\
\langle \mathbf{r}'''(s), \mathbf{v}\rangle &= 0
\end{aligned}$$

(ii). By (b) (i), we have $\langle \mathbf{r}'(s), \mathbf{v}\rangle = \langle \mathbf{r}''(s), \mathbf{v}\rangle = 0$
That is: $\mathbf{r}'(s) \perp \mathbf{v}$ and $\mathbf{r}''(s) \perp \mathbf{v}$
Therefore $\mathbf{r}'(s) \times \mathbf{r}''(s) = \lambda\,\mathbf{v}$ for some $\lambda \in \mathbb{R}$ (depend on s)
Hence $\langle \mathbf{r}'(s) \times \mathbf{r}''(s), \mathbf{r}'''(s)\rangle = \langle \lambda\,\mathbf{v}, \mathbf{r}'''(s)\rangle = 0$

(iii). By torsion formula, $\tau(s) = \dfrac{\langle \mathbf{r}'(s) \times \mathbf{r}''(s), \mathbf{r}'''(s) \rangle}{\| \mathbf{r}'(s) \times \mathbf{r}''(s) \|^2} = 0$

Candidate Performance Report

There are 28 effective attempts on this question. The average score is 8.929 out of 15. The performance of (a) is unsatisfactory while the performance of (b) is good.

Regarding (a), the most significant mistake is false recognition of question: Candidates are required to prove that if curvature $\kappa(t) = 0$, then it is a straight-line. A number of candidates showed "curvature of a straight-line is zero", which is unfortunately irrelevant. By the principle of "no any rewardable material, examiners can only give a zero score".

While examiners appreciate most candidates could point out the unit tangent vector $\mathbf{T}(t)$ is a constant vector, that is: $\mathbf{r}'(t) = \| \mathbf{r}'(t) \| \, \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^3$, they often struggles or directlu claim that $\mathbf{r}$ is a straight-line. Those responses rely in geometrical intuitions and common sense, without rigorous mathematical justification. E.g. "no direction change, therefore straigt-line". Partial-credits are given. Some candidates gives incorrect expressions like: $\mathbf{r}(t) = \mathbf{a} \| \mathbf{r}(t) \| \, \mathbf{r}(t) + \mathbf{b}$ and $\mathbf{r}(t) = \| \mathbf{r}(t) \| \, \mathbf{a}$

Very few candidates could integrate both sides from a to t, only one candidate identified $I(t) = \int_a^t \| \mathbf{r}'(t) \| dt$ as a real-valued function in t, hence successfully write $\mathbf{r}(t) = I(t) \, \mathbf{a} + \mathbf{b}$. Although no candidates justified $I(t)$ is an increasing function, responses are good enough for a 9-day short course in differential geometry. Examiners highly appreciate the effort of candidates.

Interestingly, there are few candidates (without loss of generality) assumed "$\mathbf{r}(t)$ is parametrized by arc-length". It is a reasonable assumption and an elegant approach, since it reduces the problem of $\mathbf{T}'(t) = \mathbf{0}$ to $\mathbf{r}''(s) = \mathbf{0}$. Integrate twice shows $\mathbf{r}$ is a straight-line. Several incomplete responses adopting such approach also received partial credits for the idea and clever observation.

Regarding (b), almost all candidates successfully proved (i). (Except 1 candidate scored full mark in (ii) and (iii) but skipped (i), leaving examiners astonished and impressed). A substantial number of candidates struggled in (ii). They justified $\langle \mathbf{r}'(s) \times \mathbf{r}''(s), \mathbf{r}'''(s) \rangle = 0$ by proving $\mathbf{r}' = \mathbf{r}'' = \mathbf{r}''' = \mathbf{0}$, which is impossible. They cannot deduce "$\mathbf{v}$ parallel to $\mathbf{r}'(s) \times \mathbf{r}''(s)$" from "$\langle \mathbf{r}'(s), \mathbf{v} \rangle = \langle \mathbf{r}''(s), \mathbf{v} \rangle = 0$". Actually this is an open secret since they can copy such reasoning from Question 7 (b) and (c).

The performance of (iii) is very good. Almost all candidates can use (ii) to deduce (iii) with torsion formula: $\tau(t) = \dfrac{\langle \mathbf{r}'(t) \times \mathbf{r}''(t), \mathbf{r}'''(t) \rangle}{\| \mathbf{r}'(t) \times \mathbf{r}''(t) \|^2}$. Very few candidates recalled the definition of torsion $\tau(t) = \left\langle \dfrac{d\mathbf{N}}{ds}, \mathbf{B} \right\rangle$ and tried to deduce $\mathbf{N}, \mathbf{B}$ from (ii). Such approach is logically possible but practically difficult. It is recommended to observe the guideline of question. A good question with strong discriminator should have strong logical connection between sub-parts. So the previous part is useful to the later part almost surely (with probability 1). (Please forget about pbal in question 2)

5. (15 points) This question is investigates curves on sphere

Let $\alpha(s)$ be a regular space curve with arc length parameterization. $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the unit tangent, unit normal and unit binormal to the curve respectively. Let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion of the curve. Suppose $\alpha(s)$ lies on the some sphere centred at the origin for any s, that is:

$$\|\alpha(s)\| = r \text{ for some } r \in \mathbb{R}^+$$

(a) (4 points) Differentiating with respect to s, show that

$$\langle \alpha, \alpha' \rangle = 0 \text{ and } \langle \alpha', \alpha' \rangle = -\langle \alpha'', \alpha \rangle \text{ and } \langle \alpha''', \alpha \rangle = 0.$$

(b) (3 points) Using (a) (ii) and Frenet Serret Equation, show that $\langle \alpha(s), \mathbf{N}(s) \rangle = -\dfrac{1}{\kappa(s)}$.

(c) (3 points) Using Frenet Serret Equation, show that

$$\alpha'''(s) = \frac{d}{ds}(\kappa(s)\,\mathbf{N}(s)) = -\kappa^2(s)\,\mathbf{T}(s) + \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\tau(s)\,\mathbf{B}(s).$$

(d) (i) (3 points) By considering $\langle \alpha'''(s), \alpha(s) \rangle = 0$, compute $\langle \alpha(s), \mathbf{B}(s) \rangle$

(ii) (2 points) Deduce that $\alpha(s) = -\dfrac{1}{\kappa(s)}\mathbf{N}(s) + \dfrac{\kappa'(s)}{\kappa^2(s)\tau(s)}\mathbf{B}(s)$.

(**Hints:** $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ constitutes an orthonormal basis.)

Solution

(a) Suppose $\|\alpha(s)\| = r$

$$\begin{aligned}
\langle \alpha(s), \alpha(s) \rangle &= r^2 \\
\frac{d}{ds}\langle \alpha(s), \alpha(s) \rangle &= 0 \\
2\langle \alpha'(s), \alpha(s) \rangle &= 0 \\
\langle \alpha'(s), \alpha(s) \rangle &= 0 \qquad \ldots\ldots(1) \\
\frac{d}{ds}\langle \alpha'(s), \alpha(s) \rangle &= 0 \\
\langle \alpha''(s), \alpha(s) \rangle + \langle \alpha'(s), \alpha'(s) \rangle &= 0 \\
\langle \alpha''(s), \alpha(s) \rangle &= -\langle \alpha'(s), \alpha'(s) \rangle \\
\langle \alpha''(s), \alpha(s) \rangle &= -\|\alpha'(s)\|^2 \\
\langle \alpha''(s), \alpha(s) \rangle &= -1 \qquad \ldots\ldots(2)(\text{Since } \alpha \text{ is arc-length parametrised}) \\
\frac{d}{ds}\langle \alpha''(s), \alpha(s) \rangle &= 0 \\
\langle \alpha'''(s), \alpha(s) \rangle + \langle \alpha''(s), \alpha'(s) \rangle &= 0 \\
\langle \alpha'''(s), \alpha(s) \rangle &= -\underbrace{\langle \alpha''(s), \alpha'(s) \rangle}_{\text{will be shown zero}} \qquad \ldots\ldots(3)
\end{aligned}$$

By assumption, $\alpha(s)$ is arc-lenth parametrised, hence $\langle \alpha'(s), \alpha'(s) \rangle = 1$
Differentiating gives $\langle \alpha''(s), \alpha'(s) \rangle = 0$, hence (3) implies

$$\langle \alpha'''(s), \alpha(s) \rangle = 0$$

(b) By Frenet Serret Equation, $\mathbf{T}(s) = \alpha'(s)$, hence (1) implies

$$\langle \alpha(s), \mathbf{T}(s) \rangle = 0$$

By Frenet Serret Equation, $\alpha''(s) = \mathbf{T}'(s) = \kappa(s)\,\mathbf{N}(s)$, hence (2) implies

$$\langle \alpha(s), \mathbf{N}(s) \rangle = -\frac{1}{\kappa(s)}$$

(c)

$$\begin{aligned}
\alpha'''(s) &= \frac{d}{ds}\big(\kappa(s)\,\mathbf{N}(s)\big) \\
&= \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\,\mathbf{N}'(s) \\
&= \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\big[-\kappa(s)\,\mathbf{T}(s) + \tau(s)\,\mathbf{B}(s)\big] \\
&= -\kappa(s)^2\,\mathbf{T}(s) + \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\tau(s)\,\mathbf{B}(s)
\end{aligned}$$

(d) (i).

$$\begin{aligned}
\langle \alpha'''(s), \alpha(s) \rangle &= \langle -\kappa(s)^2\,\mathbf{T}(s) + \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\tau(s)\,\mathbf{B}(s), \alpha(s) \rangle \\
0 &= -\kappa(s)^2 \langle \mathbf{T}(s), \alpha(s) \rangle + \kappa'(s) \langle \mathbf{N}(s), \alpha(s) \rangle + \kappa(s)\tau(s) \langle \mathbf{B}(s), \alpha(s) \rangle \qquad \ldots (4)
\end{aligned}$$

Substitute $\underbrace{\langle \alpha(s), \mathbf{T}(s) \rangle = 0}_{\text{consequence of (1)}}$ and $\underbrace{\langle \alpha(s), \mathbf{N}(s) \rangle = -\frac{1}{\kappa(s)}}_{\text{consequence of (2)}}$ into (4)

$$\langle \alpha(s), \mathbf{B}(s) \rangle = \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}$$

(ii). Since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ constitutes an orthonormal basis, hence

$$\alpha(s) = \underbrace{\langle \alpha(s), \mathbf{T}(s) \rangle}_{0}\,\mathbf{T}(s) + \underbrace{\langle \alpha(s), \mathbf{N}(s) \rangle}_{-\frac{1}{\kappa(s)}}\,\mathbf{N}(s) + \underbrace{\langle \alpha(s), \mathbf{B}(s) \rangle}_{\frac{\kappa'(s)}{\kappa(s)^2\tau(s)}}\,\mathbf{B}(s)$$

$$\alpha(s) = -\frac{1}{\kappa(s)}\,\mathbf{N}(s) + \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}\,\mathbf{B}(s)$$

Candidate Performance Report

There are 21 effective attempts on this question. The average score is 11.14 out of 15.

I don't have much comments on this question. Almost all candidates can finish (a) successfully. Many candidates also deduced (b). Some candidates leave their answer in terms of $\|\mathbf{r}''(s)\|$. It is accepted but it would be better if those candidates can also explain "since $\alpha(s)$ is parametrized by arc-length, therefore $\|\mathbf{r}''(s)\| = \kappa(s)$".

Concerning (c), the result is a bit extreme, that is: For those candidates who are familiar with produce rule, they often give a perfect answer by identifying $\mathbf{N}'(s) = -\kappa(s)\,\mathbf{T}(s) + \tau(s)\,\mathbf{B}(s)$. But examiners observed some candidates failed to use product rule on $\kappa(s)\,\mathbf{N}(s)$. We suspect those candidates have difficulties on differentiation of vector-valued function multiplied scalar function. They probably confused the dimension and coordinate functions.

Finally, (d) serves as a discriminator for average candidate and able candidate. It is okay to say I don't know. Notably many candidates can find the hidden information: $\langle \alpha(s), \mathbf{T}(s) \rangle = 0$, hence find the orthonormal linear combination.

Harder Part – Surface Theory (5+10 points).

Answer **ALL** True/False Questions. Answer **ONE** question in **Surface Theory** section.

General notations:

- Let $\mathbf{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a regular parametrized surface. We name the surface as M.
- Unit normal vector $\mathbf{n}$ of $\mathbf{X}$ is computed by $\mathbf{n} = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$.

Surface Theory – True/False Questions

6. (5 points) Mark each of the following statements "True" (meaning that it is a true statement) or "False" (meaning that there are counterexamples to the statement). No reasoning is required and each question carries 1 point.

   (a) A curve C lying on a regular surface M must be a regular curve.

   (b) A connected subset of a regular surface must be a regular surface.

   (c) The first fundamental form of a regular surface as a 2×2 matrix must be invertible.

   (d) Let M be a surface obtained by revolving the curve $z = f(x)$ around the z-axis. Then $(x\cos\theta, x\sin\theta, f(x))$, $0 < \theta < 2\pi$ gives a parametrization of M.

   (e) Surface area of M is given by: $\displaystyle\iint_D \sqrt{\det(I)}dA$, where I is the first fundamental form of $\mathbf{X}$ and dA is the area element of M.

7. (10 points) Let $\mathbf{a} \in \mathbb{R}^3$ be a fixed vector such that for every $(u, v) \in D$

$$\| \mathbf{X}(u, v) - \mathbf{a} \| = r \qquad \ldots (\#)$$

   (a) (1 point) Explain the geometric meaning of (#).

   (b) (3 points) Prove that $\langle \mathbf{X}_u, \mathbf{X} - \mathbf{a} \rangle = \langle \mathbf{X}_v, \mathbf{X} - \mathbf{a} \rangle = 0$.

   (c) (3 points) Prove that $\mathbf{X}(u, v) - \mathbf{a}$ is parallel to $\mathbf{X}_u \times \mathbf{X}_v$.

   (d) (3 points) Deduce that all normal vectors of $\mathbf{X}$ passes through $\mathbf{a}$.

Solution

(a) $\mathbf{X}$ is part of a sphere of radius r and centred at $\mathbf{a}$

(b)

$$\begin{aligned}
\langle \mathbf{X}(u, v) - \mathbf{a}, \mathbf{X}(u, v) - \mathbf{a} \rangle &= r^2 \\
\frac{\partial}{\partial u} \langle \mathbf{X}(u, v) - \mathbf{a}, \mathbf{X}(u, v) - \mathbf{a} \rangle &= 0 \\
\langle \mathbf{X}_u(u, v), \mathbf{X}(u, v) - \mathbf{a} \rangle + \langle \mathbf{X}(u, v) - \mathbf{a}, \mathbf{X}_u(u, v) \rangle &= 0 \\
2 \langle \mathbf{X}_u(u, v), \mathbf{X}(u, v) - \mathbf{a} \rangle &= 0 \\
\langle \mathbf{X}_u(u, v), \mathbf{X}(u, v) - \mathbf{a} \rangle &= 0
\end{aligned}$$

Similarly, we have $\langle \mathbf{X}_v(u, v), \mathbf{X}(u, v) - \mathbf{a} \rangle = 0$

(c) By (b), $\langle \mathbf{X}_u, \mathbf{X} - \mathbf{a} \rangle = \langle \mathbf{X}_v, \mathbf{X} - \mathbf{a} \rangle = 0$, therefore $\mathbf{X}_u \perp (\mathbf{X} - \mathbf{a})$ and $\mathbf{X}_v \perp (\mathbf{X} - \mathbf{a})$
Hence $\mathbf{X}_u \times \mathbf{X}_v = \mu(\mathbf{X} - a)$ for some $\mu \in \mathbb{R} \setminus \{0\}$ (depend on u, v)
That is: $\mathbf{X}(u, v) - \mathbf{a}$ is parallel to $\mathbf{X}_u \times \mathbf{X}_v$

(d) For any arbitrary point $\mathbf{X}(u_0, v_0)$ on the surface, the normal vector at $\mathbf{X}(u_0, v_0)$ is computed by $\mathbf{X}_u(u_0, v_0) \times \mathbf{X}_v(u_0, v_0)$. By (c), $\mathbf{X}(u_0, v_0) - \mathbf{a} = \lambda \mathbf{X}_u(u_0, v_0) \times \mathbf{X}_v(u_0, v_0)$, with $\lambda = \dfrac{1}{\mu} \in \mathbb{R} \setminus \{0\}$. Therefore $\mathbf{X}(u_0, v_0) - \lambda \mathbf{X}_u(u_0, v_0) \times \mathbf{X}_v(u_0, v_0) = \mathbf{a}$. That is: The normal vector at $\mathbf{X}(u_0, v_0)$ passes through $\mathbf{a}$

Candidate Performance Report

There are 28 effective attempts on this question. The average score is 6.25 out of 10.

Regarding (a), the performance is poor. Less than 5 candidates can give complete and accurate description of the geometric meaning. This part is formulated with good intentions for candidates to "do without any thinking". Since candidates can simply copy (d) of question 8, "$\mathbf{X}$ is part of sphere".

For (b), many candidates can differentiate both side to deduce $\langle \mathbf{X}_u, \mathbf{X} - \mathbf{a} \rangle = \langle \mathbf{X}_v, \mathbf{X} - \mathbf{a} \rangle = 0$. However some candidates write $\langle \mathbf{X}_u, \mathbf{X}_u - \mathbf{a} \rangle = 0$ or $\langle \mathbf{X}_u - \mathbf{a}, \mathbf{X}_u - \mathbf{a} \rangle = 0$, demonstrating failure to generalise single variable differentiation to multivariable differentiation. Moreover, since $\sqrt{\langle \mathbf{X}(u,v) - \mathbf{a}, \mathbf{X}(u,v) - \mathbf{a} \rangle}$ is difficult to differentiate, examiners recommends candidates to "square both sides before differentiating".

For (c), the performance is similar to Question 4 (b) (ii). Some candidates lack geometric intuition of $\mathbf{X}_u \times \mathbf{X}_v$. But examiners are glad to see that a substantial number of candidaates stated "$\mathbf{a} \times \mathbf{b} = \| \mathbf{a} \| \| \mathbf{b} \| \sin\theta \, \mathbf{n}$ and finished (c) successfully.

For (d), this is rather an English reading and translation problem. Candidates are required to formulate "normal vector of $\mathbf{X}$ passes through $\mathbf{a}$", using mathematical language: "Using $\mathbf{X}(u_0, v_0)$ as a starting point, can we stretch (rescale) the vector $\mathbf{X}_u \times \mathbf{X}_v$ so that the resultant vector meets point $\mathbf{a}$?"

8. (10 points) Let $\mathbf{a} \in \mathbb{R}^3$ be a fixed vector such that for every $(u, v) \in D$

$$\mathbf{X}(u, v) - \mathbf{a} = \lambda(u, v)\,\mathbf{n} \qquad \ldots(\#\#)$$

for some smooth functions $\lambda : U \subset \mathbb{R}^2 \to \mathbb{R}$

(a) (1 point) Explain the geometric meaning of (##).

(b) (2 points) Show that $\| \mathbf{X} - \mathbf{a} \|^2 = \lambda^2$.

(c) (4 points) Using the fact that $\langle \mathbf{X}_u, \mathbf{n}\rangle = \langle \mathbf{X}_v, \mathbf{n}\rangle = 0$, prove that

$$\frac{\partial}{\partial u}\lambda^2 = \frac{\partial}{\partial v}\lambda^2 = 0.$$

(d) (3 points) Deduce that $\mathbf{X}$ is part of a sphere. What is the radius of the sphere?

(a) All normal vectors of $\mathbf{X}$ passes through $\mathbf{a}$

(b) Since $\mathbf{n}$ denotes the unit normal vector, hence $\| \mathbf{n} \| = 1$, thus

$$\begin{aligned}
\mathbf{X}(u, v) - \mathbf{a} &= \lambda(u, v)\,\mathbf{n} \\
\| \mathbf{X}(u, v) - \mathbf{a} \| &= \lambda(u, v)\| \mathbf{n} \| \\
\| \mathbf{X}(u, v) - \mathbf{a} \| &= \lambda(u, v) \\
\| \mathbf{X}(u, v) - \mathbf{a} \|^2 &= \lambda(u, v)^2
\end{aligned}$$

(c) Consider $\langle \mathbf{X}(u, v) - \mathbf{a}, \mathbf{X}(u, v) - \mathbf{a}\rangle = \| \mathbf{X}(u, v) - \mathbf{a} \|^2$

$$\begin{aligned}
\frac{\partial}{\partial u}\lambda^2 &= \frac{\partial}{\partial u}\langle \mathbf{X}(u, v) - \mathbf{a}, \mathbf{X}(u, v) - \mathbf{a}\rangle \\
&= \langle \mathbf{X}_u(u, v), \mathbf{X}(u, v) - \mathbf{a}\rangle + \langle \mathbf{X}(u, v) - \mathbf{a}, \mathbf{X}_u(u, v)\rangle \\
&= 2\langle \mathbf{X}_u(u, v), \mathbf{X}(u, v) - \mathbf{a}\rangle \\
&= 2\langle \mathbf{X}_u(u, v), \lambda(u, v)\,\mathbf{n}(u, v)\rangle \\
&= 2\lambda(u, v)\langle \mathbf{X}_u(u, v), \mathbf{n}(u, v)\rangle \\
&= 0
\end{aligned}$$

Similarly, we have $\dfrac{\partial}{\partial v}\lambda^2 = 0$

(d) It suffices to show that $\lambda(u, v)^2$ is a constant. By (c), $\dfrac{\partial}{\partial u}\lambda^2 = \dfrac{\partial}{\partial v}\lambda^2 = 0$

Integrating with respect to u, we have $\displaystyle\int \frac{\partial}{\partial u}\lambda(u, v)^2 du = 0$

That is $\lambda(u, v)^2 + f(v) = 0$ for some $f : \mathbb{R} \to \mathbb{R}$ depends on v only

Partial differentiating both side with respect to v: $\dfrac{\partial}{\partial v}\lambda(u, v)^2 = \dfrac{\partial}{\partial v}f(v)$

So we have $f'(v) = 0$, which implies $f(v) = C$ for some constant independent of u, v

Therefore $\lambda(u, v) = \widetilde{C}$ for some $\widetilde{C}$ independent of u, v

By (b), $\| \mathbf{X}(u, v) - \mathbf{a} \| = \lambda$, therefore lies on a sphere with radius $\lambda = \widetilde{C}$ and centre $\mathbf{a}$

Candidate Performance Report

There are only 4 effective attempts on this question, indicating this is an unpopular question. The average score is 3.5 out of 10. The responses are often incomplete. It is inappropriate to conduct statistical analysis on such special case with very few samples.

Bonus Part (15 points). Try your best to answer the question in this part.

9. This question will investigate the tangent plane of a surface and the geometric application of the first fundamental form.

   Let S be a regular surface parameterized by $\mathbf{X}(u,v)$ for any $(u,v) \in U$. For any point $p := \mathbf{X}(u_0, v_0)$, we can define a **tangent space** T_pS of S at p by

   $$T_pS := \{\alpha\,\mathbf{X}_u(u_0,v_0) + \beta\,\mathbf{X}_v(u_0,v_0) : \alpha, \beta \in \mathbb{R}\}.$$

   (a) (5 points) Show that for any $(u_0, v_0) \in U \subseteq \mathbb{R}^2$, $\mathbf{x} = \alpha\,\mathbf{X}_u(u_0,v_0) + \beta\,\mathbf{X}_v(u_0,v_0) \in T_pS$, we have

   $$\langle \mathbf{x}, \mathbf{x} \rangle = \begin{pmatrix} \alpha & \beta \end{pmatrix} I_p \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

   where I_p denotes the first fundamental form as a 2×2 matrix evaluated at p.

   (b) (5 points) Let $\mathbf{r} : V \subseteq \mathbb{R} \to S$, be a regular parameterized curve on the surface S defined by

   $$\mathbf{r}(t) := \mathbf{X}(f(t), g(t)) \text{ for any } t \in V$$

   and let $s = s(t)$ be the arc-length of $\mathbf{r}$. Given that

   $$\mathbf{r}'(t) = f'(t)\,\mathbf{X}_u(f(t), g(t)) + g'(t)\,\mathbf{X}_v(f(t), g(t)),$$

   show that

   $$\left(\frac{ds}{dt}\right)^2 = \begin{pmatrix} f'(t) & g'(t) \end{pmatrix} I \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix}.$$

   (c) (5 points) Now let $\mathbf{Y}(u,v)$ be a regular parameterized surface with first fundamental form

   $$I = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(v) \end{pmatrix}.$$

   Let $\alpha(t) := \mathbf{Y}(\sin(\log t), \log t)$ for $t \in (1, e^\pi)$. Find the arc-length of $\alpha(t)$.

Solution. In this solution, the arguments of the function is omitted for simplicity.

(a) Let $\mathbf{x} = \alpha\,\mathbf{X}_u + \beta\,\mathbf{X}_v$, then

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{x} \rangle &= \langle \alpha\,\mathbf{X}_u + \beta\,\mathbf{X}_v, \alpha\,\mathbf{X}_u + \beta\,\mathbf{X}_v \rangle \\
&= \langle \alpha\,\mathbf{X}_u, \alpha\,\mathbf{X}_u \rangle + \langle \alpha\,\mathbf{X}_u, \beta\,\mathbf{X}_v \rangle + \langle \beta\,\mathbf{X}_v, \alpha\,\mathbf{X}_u \rangle + \langle \beta\,\mathbf{X}_v, \beta\,\mathbf{X}_v \rangle \\
&= \alpha^2 \langle \mathbf{X}_u, \mathbf{X}_u \rangle + \alpha\beta \langle \mathbf{X}_u, \mathbf{X}_v \rangle + \beta\alpha \langle \mathbf{X}_v, \mathbf{X}_u \rangle + \beta^2 \langle \mathbf{X}_v, \mathbf{X}_v \rangle \\
&= \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \alpha \langle \mathbf{X}_u, \mathbf{X}_u \rangle + \beta \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \alpha \langle \mathbf{X}_v, \mathbf{X}_u \rangle + \beta \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \beta \end{pmatrix} I \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\end{aligned}$$

(b) By multivariable chain rule, $\mathbf{r}' = f'(t)\,\mathbf{X}_u(f(t), g(t)) + g'(t)\,\mathbf{X}_v(f(t), g(t))$. Also, note that $\frac{ds}{dt} = ||r'(t)||$. We have

$$\begin{aligned}
\left(\frac{ds}{dt}\right)^2 &= ||r'(t)||^2 \\
&= \langle r'(t), r'(t) \rangle \\
&= \begin{pmatrix} f'(t) & g'(t) \end{pmatrix} I \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix}
\end{aligned}$$

(c) Using (b), we have the arc-length l of α is given by

$$\begin{aligned}
l &= \int_1^{e^\pi} \sqrt{\begin{pmatrix} \frac{\cos(\log t)}{t} & \frac{1}{t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(v) \end{pmatrix} \begin{pmatrix} \frac{\cos(\log t)}{t} \\ \frac{1}{t} \end{pmatrix}} dt \\
&= \int_1^{e^\pi} \sqrt{\frac{\cos^2(\log t)}{t^2} + \frac{\sin^2(\log t)}{t^2}} dt \\
&= \int_1^{e^\pi} \sqrt{\frac{1}{t^2}} dt \\
&= \int_1^{e^\pi} \frac{1}{t} dt \\
&= \log(e^\pi) - \log(1) \\
&= \pi
\end{aligned}$$

Remark. Actually the first form represents a **inner product** on the tangent space, so we can study the geometry (like arc-length, angle, area) of the surface using the first form. Originally I wanted to study different geometric property using the first form (this whole bonus question is just a part of the original question), but I figured that it would be too difficult.